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Stokes flow due to the sliding of a smooth plate over a slotted plate

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Abstract – The Stokes flow due to the sliding of a smooth plate over a slotted plate and a bottom fixed plate is solved by eigenfunction expansions and matching. The streamlines show recirculating bubbles near the slots. The drag depends on the void ratio and the relative positions of the three plates. It is found that the Navier condition is in general not satisfied for a slotted plate. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The resistance due to the relative parallel motion of two solid surfaces separated by a viscous fluid is a basic topic in fluid mechanics. If the plates are flat and smooth, the linear velocity profile is called Couette flow. But what if one plate is slotted in the shape of a thin 2D screen? Since the no-slip conditions are only periodically satisfied on the solid surfaces of the slotted plate, the boundary condition is somewhat similar to that applied to a porous boundary. We shall not go into the voluminous literature on rough or porous boundaries, but concentrate on the flow parallel to a slotted plate.

Smith [1] first considered a slotted plate in an infinite shear flow. Using Roscoe's [2] transform, the Stokes equation is reduced to Laplace's equation and an exact solution for a single slit is found. Smith superposed the result to obtain the flow past periodically spaced slits in the asymptotic limit of zero slit widths. Tio and Sadhal [3], using Roscoe transform, analytically solved the problem of infinite shear flow over a periodically slotted plate. They used Sneddon's [4] dual series solution and considered both transverse and parallel shear flows. Laplace and Arquis [5] numerically integrated the Navier–Stokes equations and found a relation between the average slip on the plate and the average shear, which is also the shear at infinity.

Except for the asymptotic solution of Smith [1], the above authors considered symmetrical shear flow on both sides of the slotted plate, such that the plane of the slotted plate is a streamline. Thus the shear flows on opposite sides agree on the same average slip velocity, and interactions are minimal. In this paper, we shall investigate the interactions of the two sides in more detail. After all, the slots being fluid, they can and do transmit mass and momentum across the plane of the plate.

2. Singularity near the edges

At the sharp tips of the slotted plate singularities occur which may have strong influence on the flow field. We investigate this possibility by studying the vicinity of a tip. Let the plate surface be described by $\theta = 0, 2\pi$

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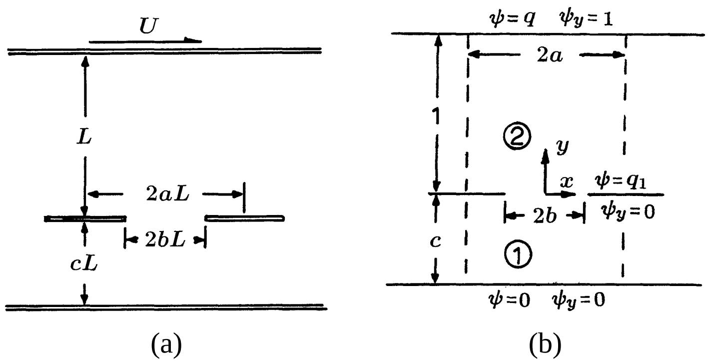


Figure 1. (a) Geometry of the plates; (b) the boundary conditions for ψ.

in the cylindrical coordinates (r,θ). The Stokes solution for the stream function ψ is a linear combination (e.g. Sherman [6]) of

$$\psi=\begin{cases} r^{\lambda}\{\sin(\lambda\theta),\cos(\lambda\theta),\sin[(\lambda-2)\theta],\cos[(\lambda-2)\theta]\}, & \lambda\neq 0,1,2,\\ \{1,\theta,\sin(2\theta),\cos(2\theta)\}, & \lambda=0,\\ r\{\sin\theta,\cos\theta,\theta\sin\theta,\theta\cos\theta\}, & \lambda=1,\\ r^2\{\sin(2\theta),\cos(2\theta),1,\theta\}, & \lambda=2.\end{cases} \tag{1}$$

For Stokes flow the velocity $(\psi_\theta/r,-\psi_r)$ are to be bounded at $r=0$, thus λ must be equal to or greater than unity. The boundary conditions are the no-slip conditions $\psi=\psi_\theta=0$ at $\theta=0,2\pi$. From equation (1) one can show that the eigenvalues are real, with $\lambda=\pm n/2$ where n is an integer. There are no non-trivial solutions that satisfy the boundary conditions for $\lambda=1$. The dominant solution thus corresponds to the eigenvalue $\lambda=3/2$, in which case we find

$$\psi=Er^{3/2}\left\{\sin\left(\frac{3}{2}\theta\right)-3\sin\left(\frac{1}{2}\theta\right)+\kappa\left[\cos\left(\frac{3}{2}\theta\right)-\cos\left(\frac{1}{2}\theta\right)\right]\right\}, \tag{2}$$

where E and κ are arbitrary constants. The flow is anti-symmetric with respect to the plane of the plate if $\kappa=0$, and thus can turn over the sharp corner. The flow is symmetric when $|\kappa|\to\infty$. For all non-zero values of κ, ψ is zero for some value of θ. Since, due to the imposed shear flow, complete anti-symmetry is not possible for our problem, we conclude a separation streamline always ending at the edge of the plate.

We find the velocities are proportional to $r^{1/2}$, which tends to zero as $r\to 0$. The stress, however, is proportional to $r^{-1/2}$. In our case this weak stress singularity does not affect the drag, which goes to zero as $r^{1/2}$. Thus the effect of the leading (and trailing) edges would be local and would not influence the gross flow field or the total flow resistance.

3. Formulation

Figure 1(a) shows the cross section. The top plate is moving transversely with velocity U while the middle slotted plate and the bottom plate are at rest. The slots have a width of $2bL$ and a period of $2aL$ where L is the distance of the slotted plate to the moving plate. For Stokes (creeping) flow the governing equation is the biharmonic equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi=0, \tag{3}$$

where the velocity components are $(\psi_y, -\psi_x)$ in the (x, y) directions. We normalize all lengths by L and the stream function by UL. *Figure 1(b)* shows the no-slip boundary conditions. Note that the value of ψ (the mass flux from bottom) is unknown on the top plate and the slotted plate. Due to periodicity, we can consider the unit cell $-a \leqslant x \leqslant a,\ -c \leqslant y \leqslant 1$. Since there is no transverse symmetry and the region is finite, the Roscoe transform fails. We shall use the method of eigenfunction expansions and matching.

We partition the cell into two regions, separated by the plane of the slotted plate. The solution for region 1 $(-c \leqslant y \leqslant 0)$ can be expressed as

$$\psi_1(x,y)=A_0(y+c)^2+\sum_1^{\infty}\cos(\alpha_n x)\big\{A_n\big[e^{\alpha_n y}-\big\langle 1+2\alpha_n(y+c)\big\rangle e^{-\alpha_n(y+2c)}\big] + B_n(y+c)\big[e^{\alpha_n y}-e^{-\alpha_n(y+2c)}\big]\big\}. \tag{4}$$

Here $\alpha_n = n\pi/a$, A_n, B_n are unknown coefficients, and ψ_1 is even in x, has period $2a$, and satisfies the no-slip boundary conditions on $y=-c$. Since there is no mean pressure gradient, the y^3 term is not included. Similarly the solution for region 2 $(0 \leqslant y \leqslant 1)$, satisfying the uniform velocity condition on $y = 1$ is

$$\psi_2(x,y)=q+y-1+C_0(y-1)^2+\sum_1^{\infty}\cos(\alpha_n x)\big\{C_n\big[e^{\alpha_n(y-2)}-\big\langle 1+2\alpha_n(y-1)\big\rangle e^{-\alpha_n y}\big] + D_n(y-1)\big[e^{\alpha_n(y-2)}-e^{-\alpha_n y}\big]\big\}, \tag{5}$$

where q, C_n, D_n are unknown coefficients. The matching conditions are the continuity of mass, velocities, shear and pressure on the boundary at $y=0$:

$$\psi_1(x,0)=\psi_2(x,0), \tag{6}$$

$$\psi_{1y}(x,0)=\psi_{2y}(x,0), \tag{7}$$

$$\psi_{1yy}(x,0)=\psi_{2yy}(x,0), \quad 0\leqslant x<b, \tag{8}$$

$$\psi_{1yyy}(x,0)=\psi_{2yyy}(x,0), \quad 0\leqslant x<b, \tag{9}$$

$$\psi_2(x,0)=q_1, \quad b<x\leqslant a, \tag{10}$$

$$\psi_{2y}(x,0)=0, \quad b<x\leqslant a. \tag{11}$$

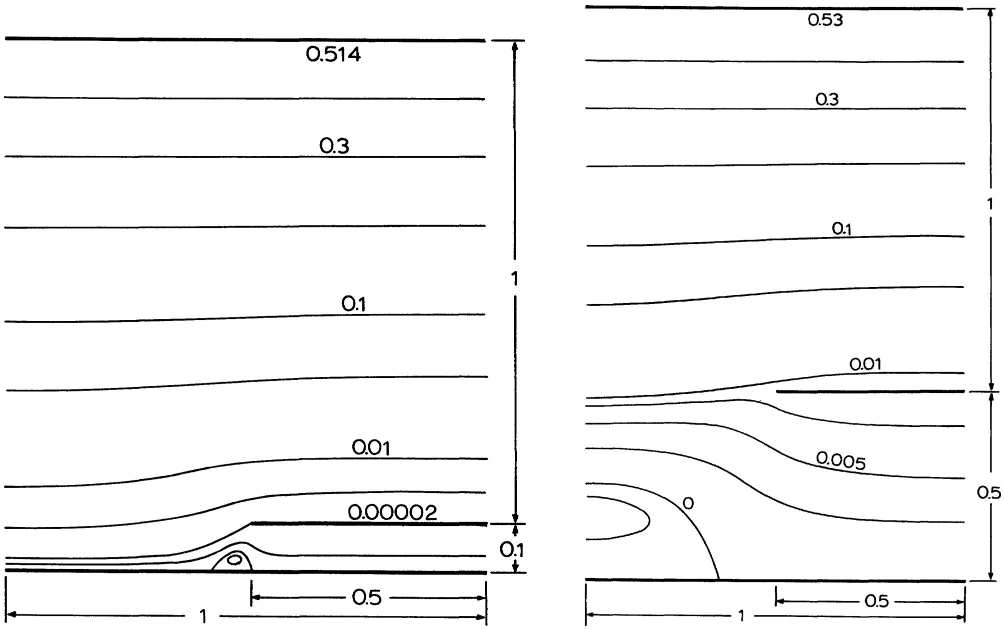
Each harmonic of equation (6), (7) can be equated to give the relation between the coefficients:

$$A_0c^2=q+C_0-1, \qquad 2A_0c=1-2C_0, \tag{12}$$

$$A_n\big[1-(1+2\alpha_nc)e^{-2\alpha_nc}\big]+B_nc\big(1-e^{-2\alpha_nc}\big)=C_n\big(e^{-2\alpha_n}-1+2\alpha_n\big)+D_n\big(1-e^{-2\alpha_n}\big), \tag{13}$$

$$A_n\alpha_n\big[1-(1-2\alpha_nc)e^{-2\alpha_nc}\big]+B_n\big[c\alpha_n\big(1+e^{-2\alpha_nc}\big)+1-e^{-2\alpha_nc}\big] = C_n\alpha_n\big(e^{-2\alpha_n}-1-2\alpha_n\big)-D_n\big[\alpha_n\big(1+e^{-2\alpha_n}\big)+1-e^{-2\alpha_n}\big]. \tag{14}$$

We solve A_0, C_0 from equation (12) in terms of q and solve A_n, B_n from equations (13), (14) in terms of C_n, D_n. These are inserted into the series in equations (4), (5) and truncated to N terms. Equations (8)–(11) are then satisfied at $N+1$ equally spaced points $x_j=(j-0.5)a/(N+1)$ with $j=1,2,\ldots,N+1$. There are



(a) (b)

Figure 2. (a) Streamlines for $a = 1,\ b = 0.5,\ c = 0.1$, values are for ψ; (b) streamlines for $a = 1,\ b = 0.5,\ c = 0.5$, values are for ψ.

$2N + 2$ linear equations and $2N + 2$ unknowns: $q, q_1, C_1, \ldots, C_N, D_1, \ldots, D_N$. These are solved by standard methods. Since our expansions are Fourier series whose convergence is assured, the accuracy can be improved by increasing N. As shown in section 1, the possible effects of the matching points missing the leading edge would be limited locally. In general, $N = 20$ would assure a three figure accuracy in stream function or drag.

4. Streamlines and drag

Figure 2(a) shows the streamlines for $a = 1,\ b = 0.5,\ c = 0.1$. Only half a period is shown since the streamlines can be mirror-reflected in the flow direction. Notice the small entrance (and exit) recirculating bubble. This bubble becomes weaker as b is increased and disappears when $b \approx a$. When b is small or c is large, two bubbles coalesce into one. *Figure 2(b)* shows the streamlines for larger c values. If we increase c and keep the other parameters fixed, the bubble eventually disappears. Note the depression of the streamlines near the slot.

The drag per length of the moving plate normalized by $U(\text{viscosity})/L$ is

$$D = \frac{1}{a}\int_0^a \psi_{2yy}(x, 1)\,\mathrm{d}x = 2C_0. \tag{15}$$

Let the void area fraction be $\beta = b/a$ and the depth-to-period ratio be $\gamma = c/a$. Thus for fixed β and γ, the geometry of the fixed plates are unchanged. The distance to the moving plate is then adjusted by the parameter a. *Figure 3* shows the drag for $\beta = 0.5$ and various different bottom plate distance γ. We see that when $\gamma = 0$, the two fixed plated coalesce, and the slots are filled. This is the regular Couette flow with $D = 1$. For $\gamma > 0$ or non-zero gap between the fixed plates, drag decreases with increased period a. Notice that when

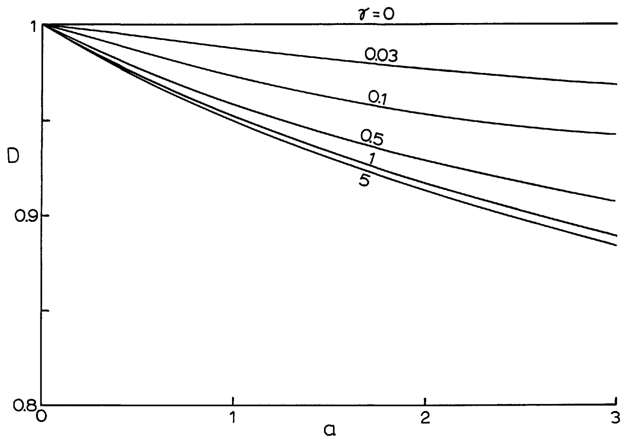


Figure 3. Drag versus a for $\beta = b/a = 0.5$ and various constants $\gamma = c/a$.

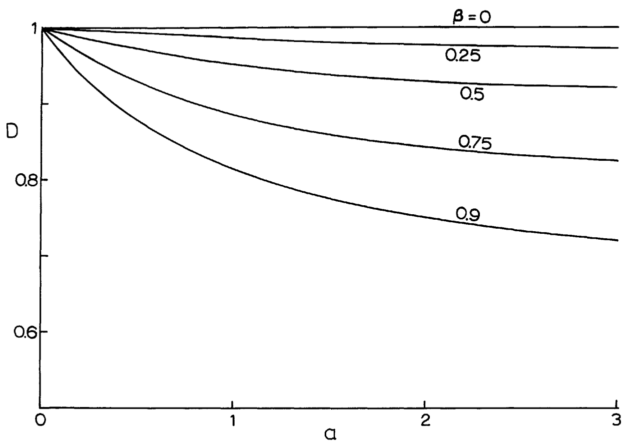


Figure 4. Drag versus a for $c = 1$ and various constants β.

$a = 0$, drag is unity for all void fractions. This means that, as the period of the slotted plate tends to zero, the slotted plate behaves like a solid plate with zero slip.

The effect of void fraction is shown in *figure 4*, where the bottom plate distance is kept at $c = 1$. We see when $\beta = 0$, that the slotted plate is continuous and thus $D = 1$. For $a \neq 0$ the drag decreases as β is increased. When $\beta = 1$, the slotted plate disappears and the flow becomes Couette flow with doubled channel width or $D = 0.5$.

For large void fraction such as $\beta = 0.9$, the number of matching points is increased to 50 in order to adequately reflect the drag.

5. Apparent slip

Beavers and Joseph [7] empirically showed that the mean slip is proportional to the mean shear at a porous boundary. This relation is called Navier condition by Laplace and Arquis [5]. Let us investigate whether the Navier condition applies to a slotted plate.

First consider the infinite, symmetric shear flows at both sides of the slotted plate. This case is amenable to Roscoe's transform and Tio and Sadhal [3] found the exact relation

$$S = \frac{2\pi}{|\ln[\cos(\pi\beta/2)]|}, \tag{16}$$

where S is the ratio of mean shear to mean slip, multiplied by the period. Laplace and Arquis [5] used numerical integration and extrapolated for infinite shear flow. As compared to the exact relation, their results are inaccurate, since they are 8% to 70% higher, with the error most pronounced for β close to zero or one. Since S is a function of β only, we conclude that the Navier condition is valid for 'symmetric, infinite' shear flow over a slotted plate.

In the present paper, we have unsymmetrical shear flow. The mean shear and mean slip can be obtained from equation (15), giving the ratio

$$S = \frac{4aC_0}{1-2C_0} = \frac{2a}{\frac{1}{D}-1}. \tag{17}$$

Thus if the Navier condition holds true, S should be constant for all constant void ratios β. From *figure 3*, for which $\beta = 0.5$, we find not only S vary with the fixed plate distance c, but also with the relative moving plate distance $1/2a$. Thus the Navier condition cannot be extended to every porous boundary, including the slotted plate studied here.

6. Discussion

The Stokes flow due to a moving plate over a slotted plate and a fixed plate is thus solved. The method of eigenfunction expansions and matching is efficient and well suited for the present problem.

Figures 2(a) and *2(b)* show that the bottom fixed plate has considerable effect on the streamlines. Most unexpected is the existence of the entrance and exit recirculating bubbles which seem to obstruct the flow. This phenomenon should be confirmed experimentally. For large c, the bubble disappears and the streamlines qualitatively agree with those of Smith [1] for a single slot in infinite shear flow.

The apparent slip or mean slip on the slotted plate can be represented by $1 - D$. This value is affected not only by the void ratio β, but also by the locations of the moving plate and the fixed plate. If we decrease the half period a while keeping β fixed, the slip becomes zero. This means for a really fine slotted plate, that the no-slip condition is appropriate for any non-zero void ratio. Such a conclusion agrees with that of Wang [8] who studied a rotating slotted cylinder.

Lastly we found that the mean-shear to mean-slip ratio at the slotted plate is not constant but depends on the conditions of the fluid on both sides. Thus the flow and boundary conditions on a slotted plate differ from those for a porous boundary or a rough boundary.

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